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Classical r-matrix, new integrable system and finite boundary condition

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Abstract. We have analysed the classical r-matrix structure of a new integrable model in two-dimensional coupled Liouville–Thirring model. Due to the non-ultralocal character of the system, a new form of $(r, -s)$ structure is obtained. It is also proved that the integrability of the model is not destroyed if non-trivial finite boundary conditions are imposed. An equation determining the form of the matrices K_+ and K_- is deduced which is a simple generalization of that of Sklyanin for the ultralocal case.

1. Introduction

The integrability of two-dimensional nonlinear systems and its relation with the existence of the 'classical r-matrix' have been investigated over a long period of time [1, 2]. The situation of an ultralocal nonlinear system is now well understood, but unfortunately a majority of the systems turn out to be non-ultralocal. In this respect some work has already been done [3] which shows that a modified form of r-matrix structure exists in such cases; this was called 'r–s' structure. Some attempts have already been made regarding the quantization of such systems [4]. Of late, new classes of non-ultralocal systems have been analysed from which further clues regarding the modified 'r–s' structure have been obtained. On the other hand, an important issue in this respect is the imposition of non-trivial boundary conditions at finite distances [5]. The preservation of the integrability requires that some special conditions are to be obeyed, even for the ultralocal systems [6]. Here, in this paper, we analyse a new integrable system called the Liouville–Thirring model in the light of classical r–s structure with finite boundary condition [7]. We observe that the non-ultralocality of the model gives rise to a new form of 'r–s' matrix. Furthermore, the rational nature of the r–matrix leads to a new type of equation for the determination of boundary matrix $K_{\pm}(\lambda)$. The involution character of the conservation laws is also discussed.

2. Formulation

The classical Liouville–Thirring model is given as;

$$\partial^2\Phi = -2J^2\exp(\Phi) \qquad i\hat{\partial}\Psi = 4J\Psi\exp(\Phi) \tag{1}$$

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where $\hat{\partial}=\gamma^{\mu}\partial_{\mu}$, $J^{\mu}=\bar{\Psi}\gamma_{\mu}\Psi$, $\Psi=\Psi^{+}\gamma^{0}$, $J^{2}=J_{\mu}J^{\mu}$, $\bar{\Psi}=\gamma^{0}\Psi$. Such a model was first studied in [8]. The Lax operator pertaining to equation (1) can be written as

$$L(x,\lambda)=\begin{pmatrix} p(x) & q(x) \\ r(x) & -p(x)\end{pmatrix}-\mathrm{i}\lambda\sigma_{3} \tag{2}$$

where

$$\begin{aligned} p(x)&=-\rho_{1}(x)\alpha_{1}(x)+\tfrac{1}{4}(\phi'(x)+\pi(x))\\ r(x)&=-\rho_{2}(x)\,\mathrm{e}^{\phi(x)}+\alpha_{1}'(x)+\rho_{1}(x)\alpha_{1}^{2}(x)-\tfrac{1}{2}\alpha_{1}(x)(\phi'(x)+\pi(x))\\ q(x)&=-\rho_{1}(x). \end{aligned} \tag{3}$$

Here primes denote differentiation with respect to the space variable x, and $\pi(x)$ is the field momentum corresponding to the field ϕ; $\pi(x)=\partial_{t}\phi$, and the following substitution has been used; $\Psi_{j}=\sqrt{\rho_{j}(t,x)}\exp[-(-1)^{j}\alpha_{j}(t,x)]$. The canonical Poisson brackets are

$$\begin{aligned} \{\pi(x),\phi(y)\}&=\delta(x-y)\\ \{\alpha_{j}(x),\rho_{k}(y)\}&=\tfrac{1}{4}(-1)^{j+1}\delta_{jk}\delta(x-y). \end{aligned} \tag{4}$$

Using equations (3) and (4) we can calculate the Poisson brackets between the elements of L,

$$\begin{aligned} \{p(x),q(y)\}&=-\tfrac{1}{4}q(x)\delta(x-y)\\ \{r(x),p(y)\}&=-\tfrac{1}{4}r(x)\delta(x-y)\\ \{p(x),r(y)\}&=\tfrac{1}{4}r(x)\delta(x-y)\\ \{q(x),p(y)\}&=\tfrac{1}{4}q(x)\delta(x-y)\\ \{p(x),p(y)\}&=-\tfrac{1}{8}\delta'(x-y)\\ \{r(x),q(y)\}&=\tfrac{1}{2}p(x)\delta(x-y)-\tfrac{1}{4}\delta'(x-y)\\ \{q(x),r(y)\}&=-\tfrac{1}{2}p(x)\delta(x-y)-\tfrac{1}{4}\delta'(x-y). \end{aligned} \tag{5}$$

Now, a simple computation leads to

$$\begin{aligned} \{L(z,\lambda)\underset{,}{\otimes}L(\omega,\mu)\}&=[S(\lambda,\mu),L(x,\lambda)\otimes I-I\otimes L(z,\mu)]\delta(z-\omega)\\ &\quad-[r(\lambda-\mu),L(x,\lambda)\otimes I+I\otimes L(z,\mu)]\delta(z-\omega)-2S(\lambda,\mu)\delta'(z-\omega)\\ &=A(z,\lambda,\omega)\delta(z-\omega)-2S\delta'(z-\omega) \end{aligned} \tag{6}$$

with

$$r(\lambda,\mu)=\tfrac{1}{8}\,(\lambda+\mu)/(\lambda-\mu)P \qquad S(\lambda,\mu)=\tfrac{1}{16}(2P-I) \tag{7}$$

where P is the permutation matrix. Here $\underset{,}{\otimes}$ denotes the Poisson bracket between the elements of L. Now to pass over to the case of transition matrix T, we should remember that in the case of non-ultralocal systems one should be careful about the nature of end-points occurring in the definition of T and the end-points in the expression of $\{T\underset{,}{\otimes}T\}$ should not coincide. Now the transition matrix is defined as the solution of the equation

$$\Psi_{x}=L(x,\lambda)\Psi \tag{8}$$

in an interval $a \leqslant x \leqslant b$ of the x-axis with suitable conditions. It may also be defined as the parallel transport operator from y to x along the x-axis (at fixed time t):

$$T(x, y, \lambda) = \exp\left[\int_y^x L(Z, \lambda)\, dz\right]. \tag{9}$$

It satisfies

$$\begin{aligned} \frac{\partial}{\partial x} T(x, y, \lambda) &= -L(x, \lambda) T(x, y, \lambda) \\ \frac{\partial}{\partial y} T(x, y, \lambda) &= T(x, y, \lambda) L(y, \lambda) \end{aligned} \tag{10}$$

along with the initial condition $T(x, x, \lambda) = I$.

An important property of T that will be useful in the following is

$$\begin{aligned} &T(x, y, \lambda) T(y, z, \lambda) = T(x, z, \lambda) \\ &T(x, y, \lambda)^{-1} = T(y, x, \lambda). \end{aligned} \tag{11}$$

Moreover, under standard boundary conditions on the fields at spatial infinity,

$$T(\lambda) = \lim_{\substack{x \to +\infty \\ y \to -\infty}} T(x, y, \lambda) \tag{12}$$

exists and is called the monodromy matrix. If $T(x, y, \lambda)$ denotes such a transition matrix, then it can be shown that

$$\begin{aligned} &\{T, (x, y, \lambda) \underset{,}{\otimes} T(u, v, \mu)\} \\ &\quad = \int_y^x dz \int_v^u d\omega\ \varepsilon(x-y)\varepsilon(u-v)\chi(x; x, y)\chi(\omega; u, v) T(x, z, \lambda) \\ &\quad \otimes T(u, \omega, \mu)\{L(z, \lambda) \underset{,}{\otimes} L(\omega, \mu)\} T(z, y, \lambda) \otimes T(\omega, v, \mu) \end{aligned} \tag{13}$$

where

$$\varepsilon(x-y) = \begin{matrix} 1 & \text{when } x > y \\ 0 & \text{when } x = y \\ -1 & \text{when } x < y \end{matrix}$$

$$\chi(z, x, y) = \begin{cases} \alpha & \text{for } z = \min(x, y) \\ 1 & \min(x, y) < z < \max(x, y) \\ \beta & \text{for } z = \max(x, y) \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

To compute $\{T \underset{,}{\otimes} T\}$ we proceed by calculating $\{T \underset{,}{\otimes} L\}$. By definition [2]

$$\begin{aligned} \{T(x, y, \lambda) \underset{,}{\otimes} L(\omega, \mu)\} &= \int_y^x dz\ \varepsilon(x-y)\chi(z; x, y)(T(x, z, \lambda) \otimes I) \\ &\quad \times \{L(z, \lambda) \underset{,}{\otimes} L(\omega, \mu)\}(T(z, y, \lambda) \otimes I) \\ &= \int_y^x dz\ \varepsilon(x-y)\chi(z; x, y)(T(x, z, \lambda) \otimes I) \\ &\quad \times [A(z, \lambda, \mu)\delta(z-\omega) - 2S\partial_z(z-\omega)](T(z, y, \lambda) \otimes I). \end{aligned} \tag{15}$$

Considering the $\delta'(z-\omega)$ part, let

$$t_1=-2\int_y^x \mathrm{d}z\ \varepsilon(x-y)\chi(z;x,y)(T(x,z,\lambda)\otimes I)S(T(z,y,\lambda)\otimes I)\frac{\partial}{\partial z}\delta(z-\omega)$$

$$=2\frac{\partial}{\partial z}\{\varepsilon(x-y)\chi(z;x,y)(T(x,z,\lambda)\otimes I)S(T(z,y,\lambda)\otimes I)\}|_{z=\omega}$$

$$=2\bigg[\left(\frac{\partial}{\partial z}\{\varepsilon(x-y)\chi(z;x,y)\}\right)(T(x,z,\lambda)\otimes I)S(T(z,y,\lambda)\otimes I)$$

$$+\varepsilon(x-y)\chi(z;x,y)\left(\frac{\partial}{\partial z}T(x,z,\lambda)\otimes I\right)S(T(z,y,\lambda)\otimes I)$$

$$+\varepsilon(x-y)\chi(z;x,y)(T(x,z,\lambda)\otimes I)S\left(\frac{\partial}{\partial z}T(z,y,\lambda)\otimes I\right)\bigg].$$

Note that

$$\varepsilon(x-y)\frac{\partial}{\partial z}\chi(z,x,y)=\varepsilon(x-y)[\delta(z-\min(x,y))-\delta(z-\max(x,y))$$

whence we get, on simplification,

$$t_1=2(\delta(\omega-y)-\delta(\omega-x))(T(x,\omega,\lambda)\otimes I)S(T(\omega,y,\lambda)\otimes I)$$
$$-2\varepsilon(x-y)\chi(\omega,x,y)(T(x,\omega,\lambda)\otimes I)[S,L(\omega,\lambda)\otimes I](T(\omega,y,\lambda)\otimes I). \quad (16)$$

Then, if we call the $\delta(z-\omega)$ term t_2, we get

$$\{T(x,y,\lambda)\underset{,}{\otimes}L(\omega,\mu)\}=t_2+t_1 \quad (17)$$

$$=-2(\delta(\omega-x)-\delta(\omega-y))(T(x,\omega,\lambda)\otimes I)S(T(\omega,y,\lambda)\otimes I)$$
$$-\varepsilon(x-y)\chi(\omega;x,y)(T(x,\omega,\lambda)\otimes I)$$
$$\times[r(\lambda,\mu)+S,L(\omega,\lambda)\otimes I+I\otimes L(\omega,\mu)](T(\omega,y,\lambda)\otimes I). \quad (18)$$

In terms of this expression of the Poisson bracket we can express $\{T(x,y,\lambda)\otimes T(u,v,\mu)\}$ as

$$\{T(x,y,\lambda)\underset{,}{\otimes}T(u,v,\mu)\}=\int_v^u \mathrm{d}\omega\ \varepsilon(u-v)\chi(\omega,u,v)(I\otimes T(u,\omega,\mu))$$
$$\times\{T(x,y,\lambda)\underset{,}{\otimes}L(\omega,\mu)\}(I\otimes T(\omega,v,\mu)) \quad (19)$$

whence, using equation (10) and integrating over the δ-function we get

$$\{T(x,y,\lambda)\underset{,}{\otimes}T(u,v,\mu)\}=-2\varepsilon(u-v)\chi(\omega;u,v)(T(x,\omega,\lambda)\otimes T(u,\omega,\mu)S$$
$$\times(T(\omega,y,\lambda)\otimes T(\omega,v,\mu))|_{\omega=y}^{\omega=x}$$
$$-\int_v^u \mathrm{d}\omega\ \varepsilon(x-y)\varepsilon(u-v)\chi(\omega;x,y)\chi(\omega,u,v)$$
$$\times(T(x,\omega,\lambda)\otimes T(u,\omega,\mu))[r+S,L(\omega,\lambda)\otimes I+I\otimes L(\omega,\mu)]$$
$$\times(T(\omega,y,\lambda)\otimes T(\omega,v,\mu)). \quad (20)$$

Using equation (10) one can rewrite the integrand of the last term as follows:

$$= -\int_v^u \mathrm{d}\omega\ \varepsilon(x-y)\varepsilon(u-v)\chi(\omega, x, y)\chi(\omega, u, v)$$
$$\times (T(x, \omega, \lambda)\otimes T(u, \omega, \mu))[r+S, L(\omega, \lambda)\otimes I + I\otimes L(\omega, \mu)]$$
$$\times (T(\omega, y, \lambda)\otimes T(\omega, v, \mu))$$
$$= \int_v^u \mathrm{d}\omega\ \varepsilon(x-y)\varepsilon(u-v)\chi(\omega; x, y)\chi(\omega, u, v)\frac{\partial}{\partial\omega}$$
$$\times [(T(x, \omega, \lambda)\otimes T(u, \omega, \mu))(r+S)(T(\omega, y, \lambda)\otimes T(\omega, v, \mu))]. \tag{21}$$

Integrating by parts,

$$= \varepsilon(x-y)\varepsilon(u-v)\chi(\omega; x, y)\chi(\omega, u, v)$$
$$\times (T(x, \omega, \lambda)\otimes T(u, \omega, \mu))(r+S)(T(\omega, y, \lambda)\otimes T(\omega, v, \mu))|_{\omega=v}^{\omega=u}$$
$$-\int_v^u \mathrm{d}\omega\ [(\delta(\omega-y)-\delta(\omega-x))\varepsilon(u-v)\chi(\omega, u, v)$$
$$+\varepsilon(x-y)\chi(\omega, x, y)(\delta(\omega-v)-\delta(\omega-u))(T(x, \omega, \lambda)\otimes T(u, \omega, \mu))$$
$$\times (r+S)(T(\omega, y, \lambda)\otimes T(\omega, v, \mu)). \tag{22}$$

The last integral is easy to evaluate, so we get

$$\{T(x, y, \lambda) \underset{,}{\otimes} T(u, v, \mu)\}$$
$$= -2\varepsilon(u-v)\chi(\omega; u, v)(T(x, \omega, \lambda)\otimes T(u, \omega, \lambda))$$
$$\times S(T(\omega, y, \lambda)\otimes T(\omega, v, \mu))|_{\omega=y}^{\omega=x}$$
$$+\varepsilon(x-y)\varepsilon(u-v)\chi(\omega; x, y)\chi(\omega; u, v)$$
$$\times (T(x, \omega, \lambda)\otimes T(u, \omega, \mu))(r+S)(T(\omega, y, \lambda)\otimes T(\omega, v, \mu))|_{\omega=v}^{\omega=u}$$
$$+\varepsilon(u-v)\chi(\omega; u, v)(T(x, \omega, \lambda)\otimes T(u, \omega, \mu)))(r+S)$$
$$\times (T(\omega, y, \lambda)\otimes T(\omega, v, \mu))|_{\omega=y}^{\omega=x}$$
$$+\varepsilon(x-y)\chi(\omega; x, y)(T(x, \omega, \lambda)\otimes T(u, \omega, \mu))(r+S)$$
$$\times (T(\omega, y, \lambda)\otimes T(\omega, v, \mu))|_{\omega=v}^{\omega=u}$$
$$= \varepsilon(u-v)\chi(\omega; u, v)(T(x, \omega, \lambda)\otimes T(u, \omega, \mu))(r-S)$$
$$\times (T(\omega, y, \lambda)\otimes T(\omega, v, \mu))|_{\omega=y}^{\omega=x}$$
$$+\varepsilon(x-y)\chi(\omega; x, y)(T(x, \omega, \lambda)\otimes T(u, \omega, \mu))(r+S)$$
$$\times (T(\omega, y, \lambda)\otimes T(\omega, v, \mu))|_{\omega=v}^{\omega=u}$$
$$+\varepsilon(x-y)\varepsilon(u-v)\chi(\omega; x, y)\chi(\omega; u, v)$$
$$\times (T(x, \omega, \lambda)\otimes T(u, \omega, \mu))(r+S)(T(\omega, y, \lambda)\otimes T(\omega, v, \mu))|_{\omega=v}^{\omega=u}. \tag{23}$$

It is interesting to note that the matrix 'S' is independent of (λ, μ), and r is a function of (λ, μ) which depends on both $\lambda+\mu$ and $\lambda-\mu$, but has the same simple pole structure as in the ultralocal case.

In accordance with our previous observation, we have kept the points (u, v) and (x, y) all different. If we now assume that they are ordered in a specific way, then equation (23) leads to:

(i) $v<y<x<u$, then

$$\{T(x, y, \lambda) \underset{,}{\otimes} T(u, v, \mu)\} = \varepsilon(u-v)\chi(\omega; u, v)T(x, \omega, \lambda)\otimes T(u, \omega, \mu)(r-S) \times T(\omega, y, \lambda)\otimes T(\omega, y, \mu)|_{\omega=y}^{\omega=x} \tag{24}$$

(ii) $y<v<u<x$, then

$$\{T(x, y, \lambda) \underset{,}{\otimes} T(u, v, \mu)\} = \varepsilon(x-y)\chi(\omega; x, y)(T(x, \omega, \lambda)\otimes T(u, \omega, \mu))(r+S) \times (T(\omega, y, \lambda)\otimes T(\omega, v, \mu))||_{\omega=v}^{\omega=u}. \tag{25}$$

If in equations (24) and (25) we now take the limit $u\to x$, $v\to y$, then we get respectively

$$\{T(x, y, \lambda) \underset{,}{\otimes} T(x, y, \mu)\} = [r-S, T(x, y, \lambda)\otimes T(x, y, \mu)] \tag{26}$$

$$\{T(x, y, \lambda) \underset{,}{\otimes} T(x, y, \mu)\} = [r+S, T(x, y, \lambda)\otimes T(x, y, \mu)]. \tag{27}$$

So, as per the prescription of reference [3], we take the average of (16) and (17) to finally obtain,

$$\{T(x, y, \lambda) \underset{,}{\otimes} T(x, y, \mu)\} = [r, T(x, y, \lambda)\otimes T(x, y, \mu)] \tag{28}$$

so that the monodromy matrix obeys the same Poisson bracket algebra as in the ultralocal case.

3. Finite boundary condition

Of late, several attempts have been made to impose non-trivial finite boundary conditions on integrable system without destroying the property of integrability. A unique prescription was given by Sklyanin for the case of the usual antisymmetric r-matrix of the ultralocal case. For the ultralocal integrable system Sklyanin has shown that explicit boundary conditions of the Dirichlet or Neumann type can be imposed on the nonlinear fields through the explicit use of the space and time part of the Lax operator. On the other hand, since the quantum inverse scattering can be connected with the scattering matrix, it has been shown by Cherednik [9] that such boundary conditions in the space variables can be thought of as scattering from a fixed wall at the end of the axis. It was also demonstrated that in the general case it is best introduced through a redefinition of the transition matrix, as follows. Here we show that it is possible to deduce a general condition on the boundary matrices $K_{\pm}(\lambda)$, even for our non-ultralocal case with 'r–s' structure. Let us denote by $F(x, y, \lambda)$ the matrix

$$F(x, y, \lambda) = T(x, y, \lambda)K_{-}(\lambda)T^{-1}(x, y, \lambda^{-1}) \tag{29}$$

and compute the Poisson bracket between two F's:

$$\begin{aligned}
&\{F(x,y,\lambda)\otimes F(x,y,\mu)\}\\
&\quad=\{T(x,y,\lambda)\underset{,}{\otimes}T(x,y,\mu)\}K_-(\lambda)T^{-1}(x,y,\lambda^{-1})\otimes K_-(\mu)T^{-1}(x,y,\mu^{-1})\\
&\qquad+(I\otimes T(x,y,\mu))K_-(\mu)\{T(x,y,\lambda)\underset{,}{\otimes}T^{-1}(x,y,\mu^{-1})\}\\
&\qquad\times K_-(\lambda)T^{-1}(x,y,\lambda^{-1})\otimes I+T(x,y,\lambda)\\
&\qquad\times K_-(\lambda)\otimes I\{T(x,y,\lambda^{-1})\underset{,}{\otimes}T(x,y,\mu)\}I\otimes K_-(\mu)T^{-1}(x,y,\mu^{-1})\\
&\qquad+T(x,y,\lambda)K_-(\lambda)\otimes T(x,y,\lambda)\\
&\qquad\times K_-(\mu)\{T^{-1}(x,y,\lambda^{-1})\underset{,}{\otimes}T^{-1}(x,y,\mu^{-1})\}.
\end{aligned}\tag{30}$$

We now evaluate each term separately:

$$\begin{aligned}
&\{T(x,y,\lambda)\otimes T^{-1}(x,y,\mu^{-1})\}\\
&\quad=-(I\otimes T^{-1}(x,y,\mu^{-1})r(\lambda,\mu^{-1})(T(x,y,\lambda)\otimes I)\\
&\qquad+(T(x,y,\lambda)\otimes I)r(\lambda,\mu^{-1})(I\otimes T^{-1}(x,y,\mu^{-1})\\
&\{T^{-1}(x,y,\lambda^{-1})\underset{,}{\otimes}T(x,y,\mu)\}\\
&\quad=-(T^{-1}(x,y,\lambda^{-1})\otimes I)r(\lambda^{-1},\mu)(I\otimes T(x,y,\mu))\\
&\qquad+(I\otimes T(x,y,\mu))r(\lambda^{-1},\mu)(T^{-1}(x,y,\lambda^{-1})\otimes I)\\
&\{T^{-1}(x,y,\lambda^{-1})\underset{,}{\otimes}T^{-1}(x,y,\mu^{-1})\}=-[r(\lambda^{-1},\mu^{-1}),\,T^{-1}(x,y,\lambda^{-1})T^{-1}(x,y,\mu^{-1})].
\end{aligned}\tag{31}$$

Substituting in equation (30) we get

$$\begin{aligned}
&\{F(x,y,\lambda)\underset{,}{\otimes}F(x,y,\mu)\}\\
&\quad=[F(x,y,\lambda)\otimes F(x,y,\mu)r(\lambda^{-1},\mu^{-1})+r(\lambda,\mu)F(x,y,\lambda)\otimes F(x,y,\mu)]\\
&\qquad-[1\otimes F(x,y,\mu)r(\lambda,\mu^{-1})F(x,y,\lambda)\otimes I-F(x,y,\lambda)\otimes Ir(\lambda^{-1},\mu)\\
&\qquad\times(I\otimes F(x,y,\mu))]+\Delta
\end{aligned}\tag{32}$$

where Δ stands for

$$\begin{aligned}
T(x,y,\lambda)\otimes T(x,y,\mu)[&K_-^1(\lambda)r(\lambda^{-1},\mu)K_-^2(\mu)+K_-^2(\mu)r(\lambda,\mu^{-1})K_-^1(\lambda)\\
&-r(\lambda,\mu)K_-^1(\lambda)K_-^2(\mu)-K_-^1(\lambda)K_-^2(\mu)r(\lambda^{-1},\mu^{-1})].
\end{aligned}\tag{33}$$

$$K_-^1(\lambda)=K_-(\lambda)\otimes I\qquad\text{and}\qquad K_A^2(\mu)=I\otimes K_-(\mu).$$

So, from the above expression it is clear that the Poisson bracket $\{F(x,y,\lambda)\underset{,}{\otimes}F(x,y,\mu)\}$ can be expressed solely in terms of F's if and only if $\Delta=0$, whence we get the equation for the boundary matrices once the r-matrix is known:

$$r(\lambda,\mu)K_-^1(\mu)+K_-^1(\lambda)K_-^2(\mu)r(\lambda^{-1},\mu^{-1})=K_-^1(\lambda)r(\lambda^{-1},\mu)K_-^2(\mu)+K_-^2(\mu)r(\lambda,\mu^{-1})K_-^1(\mu).\tag{34}$$

Now, to prove that we still have an infinite number of commuting conserved quantities we set

$$G(x,y,\lambda)=F(x,y,\lambda)K_+(\lambda)\tag{35}$$

so that new conserved quantities are generated by

$$t(x, y, \lambda) = \mathrm{tr}(G(x, y, \lambda) \tag{36}$$

where $K_+(\lambda)$ is the boundary matrix at the other end, obeying the same set of conditions. We get, at once,

$$\begin{aligned}\{G(x, y, \lambda) \underset{,}{\otimes} G(x, y, \mu)\} &= \{F(x, y, \lambda) \underset{,}{\otimes} F(x, y, \mu)\} K_+(\lambda) \otimes K_+(\mu)\\ &= [F(x, y, \lambda) \otimes F(x, y, \mu) r(\lambda^{-1}, \mu^{-1}) + r(\lambda, \mu) F(x, y, \lambda) \otimes F(x, y, \mu)\\ &\quad - I \otimes F(x, y, \mu) r(\lambda, \mu^{-1}) F(x, y, \lambda) \otimes I - F(x, y, \lambda) \otimes I r(\lambda^{-1}, \mu)\\ &\quad \times I \otimes F(x, y, \mu)] K_+(\lambda) \otimes K_+(\mu).\end{aligned} \tag{37}$$

Since K_+ satisfies the same equation as K_- we can reduce the above expression to the form

$$\begin{aligned}&[r(\lambda, \mu), F(x, y, \lambda) K_+(\lambda) \otimes F(x, y, \mu) K_+(\mu)]\\ &\quad + G(x, y, \lambda) \otimes F(x, y, \mu) r(\lambda, \mu^{-1}) K_+^2(\mu) + F(x, y, \lambda) \otimes G(x, y, \mu)\\ &\quad \times r(\lambda^{-1}, \mu) K_+^1(\mu) - F^2(x, y, \mu) r(\lambda, \mu^{-1}) G(x, y, \lambda) \otimes K_+^2(\mu)\\ &\quad - F^1(x, y, \mu) r(\lambda^{-1}, \mu) K_+^1(\lambda) \otimes G(x, y, \mu).\end{aligned} \tag{38}$$

Taking the trace on both sides of equation (38),

$$\begin{aligned}&\{t(x, y, \lambda), t(x, y, \mu)\}\\ &\quad = 0 + Tr(r(\lambda, \mu^{-1}) K_+^2(\mu) G(x, y, \lambda) \otimes F(x, y, \mu))\\ &\qquad + \mathrm{Tr}(r(\lambda^{-1}, \mu) K_+(\lambda) F(x, y, \lambda) \otimes G(x, y, \mu))\\ &\qquad - \mathrm{Tr}(r(\lambda, \mu^{-1}) G(x, y, \lambda) \otimes K_+(\mu) F(x, y, \mu))\\ &\qquad - \mathrm{Tr}(r(\lambda^{-1}, \mu) K_+(\lambda) F(x, y, \lambda) \otimes G(x, y, \mu) = 0.\end{aligned} \tag{39}$$

It may be interesting to note that in equation (34) for the K's we have not assumed any symmetry property. So, even in the non-ultralocal case, with the 'r–s' structure we can have a commuting set of integrals of motions, and so the integrability is preserved.

4. Explicit form of $K_\pm(\lambda)$ and conserved quantities

After the preceding general study we now proceed to give the explicit form of the boundary matrices and some conserved quantities. Assuming, as in [6], that the $K_\pm$ are diagonal, equation (34) leads to the following, with the r and s matrices as given in [7].

$$\frac{\lambda+\mu}{\lambda-\mu}(K_{11}(\lambda)K_{22}(\mu) - K_{11}(\mu)K_{22}(\lambda)) = \frac{\lambda\mu+1}{\lambda\mu-1}(K_{11}(\lambda)K_{11}(\mu) - K_{22}(\lambda)K_{22}(\mu) \tag{40}$$

which can be solved for K_{ij}.

On the other hand, the Lax equation

$$\Phi_x - L\Phi \tag{41}$$

where $\Phi=(\Phi_1, \Phi_2)'$ and L is given by (2), can be converted to the Ricatti form

$$\hat{\Phi}_x = p(x) + \frac{1}{2i\lambda}\left[-qr - p^2 + \hat{\Phi}_x^2 + q\left(\frac{\hat{\Phi}_x}{q}\right)_x - q\left(\frac{p}{q}\right)_x\right] \tag{42}$$

with

$$\hat{\Phi} = \Phi\, e^{-i\lambda x}.$$

Substituting a Laurent expansion of $\hat{\Phi}$,

$$\hat{\Phi} = \sum_{n=0}^{\infty} \frac{\hat{\Phi}_{nx}}{(2i\lambda)^n} \tag{43}$$

we get

$$\hat{\Phi}_{0x} = p(x)$$

$$\hat{\Phi}_{1x} = -qr$$

$$\hat{\Phi}_{2x} = -(2pqr + qr_x)$$

and in general

$$\hat{\Phi}_{n+1,x} = \sum_{l=0}^{n} \hat{\Phi}_{lx}\hat{\Phi}_{n-l,x} + q\left(\frac{\hat{\Phi}_{nx}}{q}\right)_x \qquad n \geqslant 1. \tag{44}$$

On the other hand, the conserved functional $a(\lambda)$ is given by

$$\ln a(\lambda) = \lim_{x\to\infty} \sum_{n=0}^{\infty} \frac{\hat{\Phi}_n}{(2i\lambda)^n} = \sum_{n=0}^{\infty} \frac{C_n}{(2i\lambda)^n} \tag{45}$$

whence we get

$$C_0 = \int_{-\infty}^{\infty} p(x)\, dx$$

$$C_1 = -\int_{-\infty}^{\infty} q(z)r(z)\, dz \tag{46}$$

$$C_2 = -\int_{-\infty}^{\infty} (2pqr + qr_z)(z)\, dz$$

etc. Using the basic Poisson brackets given in equation (5), we immediately find that

$$\{C_0, C_1\} = 0 \qquad \{C_0, C_2\} = 0 \qquad \{C_0, C_3\} = 0$$

and in general

$$\{C_i, C_j\} = 0. \tag{47}$$

5. Discussion

In the above analysis we have discussed some important properties of a new integrable model in two dimension which is non-ultralocal and its Poisson structure gives rise to

$(r, -s)$ matrix. It may be noted that the 's' part is constant but the 'r' part is a rational function of the spectral parameter, with a simple pole structure, as in the ultralocal case. It is proved that even in such a situation one can impose non-trivial boundary conditions at finite distances and preserve integrability.

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